Steven Rosenberg, Michael Spillane, and Daniel B. Wulf

Heron Triangles and Moduli Spaces

We teach triangle congruency every fall, proving and using the big five theorems—SSS, ASA, SAS, AAS, and HL. Preparing to teach this a few years ago, a colleague, Chuck Garabedian, wondered aloud if triangles with the same area and perimeter are congruent. Before you read any further, either prove this or find a counterexample.

This is not an elementary question for a high school student. Even locating a specific counterexample that a student would find believable is not particularly straightforward. (The cat is out of the bag: Counterexamples exist.) So instead of hunting for one counterexample, we set out to explore the set of all triangles with a fixed area and perimeter. Understanding the geometric structure of the set of all triangles, which we pretentiously call "the geometry of geometry," led us to a year-long investigation in parts of geometry, algebra, and number theory—all related to Chuck's original question. (The Geometer's Sketchpad [GSP] files for all figures are available at www.focusonmath.org/FOM/resources/publications/HeronTrianglesModuli.gsp. Several files contain animated components, allowing readers to explore properties of the figures in depth.)

This department focuses on mathematics content that appeals to secondary school teachers. It provides a forum that allows classroom teachers to share the mathematics from their work with students, their classroom investigations and projects, and their other experiences. We encourage submissions that pose and solve a novel or interesting mathematics problem, expand on connections among different mathematical topics, present a general method for describing a mathematical notion or solving a class of problems, elaborate on new insights into familiar secondary school mathematics, or leave the reader with a mathematical idea to expand. Send submissions to "Delving Deeper" by accessing **mt.msubmit.net**.

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Editors' note: This article appealed to us for several reasons. The first is that it shows once again how simple-sounding questions about topics in high school mathematics often lead to investigations that take one to the frontiers of mathematical research.

The article also shows how collaborations between mathematicians and teachers can be productive and satisfying work for all involved. Rosenberg, Spillane, and Wulf have been working together in a study group (as part of the Focus on Mathematics partnership in the Boston area) for several years and have formed an ongoing research team that is continually finding new problems to consider.

The article also makes use of a kind of mathematical representation that can be applied to many questions that arise in high school mathematics. The authors parameterize the triangles they investigate with a pair of numbers (the area and one side length), and, thinking of these pairs as coordinates for points, they create a *moduli space* for all triangles that share certain attributes. This idea of parameterizing phenomena with coordinates is useful in many areas of mathematics.

The authors took several years to figure out and polish some of the ideas presented in this article, so readers may not want to try to digest the entire development in one sitting. We read the article through several times before our thoughts started to gel. Although the ideas are subtle, the machinery required lies squarely within the high school curriculum. As we sometimes do with dense articles, we will intersperse editorial notes throughout, filling in some details that caused us to reach for a pencil. Adventurous readers are encouraged to ignore our notes until they have wrestled with the details for themselves.

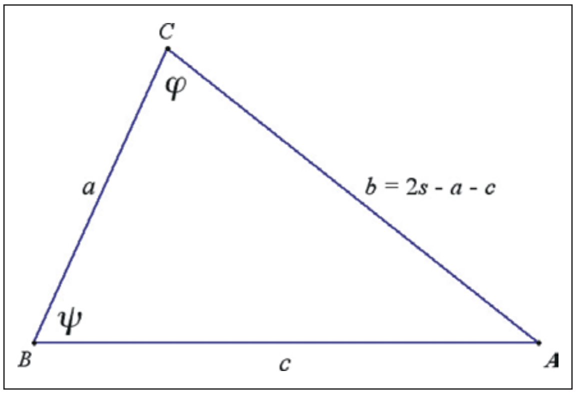

Fig. 1 The starting triangle with semiperimeter s

GEOMETRY TO ALGEBRA

We first approached the problem by converting the geometry to algebra. We call the fixed area A, the fixed semiperimeter (half the perimeter) s, and the variable lengths of the sides of the triangle a, b, and c. (See **fig. 1**.) We can eliminate some of the variables using Heron's formula

$$A^2 = s(s-c)(s-a)(s-b) \text{ and } b = 2s - a - c$$

to get

$$A^2 = s(s-c)(s-a)(s-(2s-a-c)).$$

Solving for a by the quadratic formula gives

$$a = \frac{2s-c}{2} \pm \sqrt{\frac{c^2}{4} - \frac{A^2}{s(s-c)}}. \qquad (1.1)$$

This looks like a mess, but it tells us something: In order for a triangle to exist, we must have

$$\frac{c^2}{4} - \frac{A^2}{s(s-c)} \geq 0,$$

or

$$f(c) = -sc^3 + s^2c^2 - 4A^2 \geq 0.$$

(See **fig. 2**.) By a little calculus, the "evil cubic" $f(c)$ has a local maximum at

$$\left(\frac{2}{3}s,\ \frac{4}{27}s^4 - 4A^2\right),$$

so for any triangle, we must have

$$A^2 \leq \frac{s^4}{27}. \qquad (1.2)$$

It is easy to check that an equilateral triangle has $A^2 = s^4/27$, so this gives an elementary calculus proof that an equilateral triangle maximizes area among triangles with fixed perimeter. (The usual proof involves multivariable calculus; for a clever proof without calculus, apply the arithmetic-geometric mean inequality

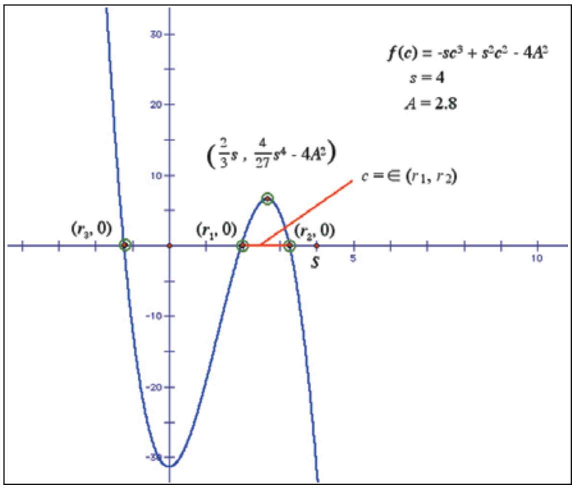

Fig. 2 The "evil cubic." A triangle with base length c and area A exists when c is between r_1 and r_2.

$$xyz \leq \frac{(x+y+z)^3}{27}$$

to $x = s-a$, $y = s-b$, $z = s-c$ in Heron's formula.)

Editors' note: The arithmetic-geometric mean inequality is usually stated in high school texts for two variables: If x and y are non-negative numbers, their geometric mean is never larger than their arithmetic mean, and equality holds only when $x = y$. In symbols,

$$\sqrt{xy} \leq \frac{(x+y)}{2}$$

with equality only when $x = y$. In fact, there is a similar inequality for any number of variables. For three variables, it says that

$$\sqrt[3]{xyz} \leq \frac{(x+y+z)}{3}$$

with equality only if $x = y = z$. This implies the inequality stated by the authors, and it also implies that if $x + y + z$ is constant, xyz is maximized when $x = y = z$. And if $x = s - a$, $y = s - b$, and $z = s - c$, then

$$x + y + z = s - a + s - b + s - c = 3s - (a + b + c) = 3s - 2s = s,$$

a constant. Since

$$\frac{A^2}{s} = (s-a)(s-b)(s-c),$$

A^2/s is largest (and hence A is largest) when $s - a = s - b = s - c$, that is, when $a = b = c$.

By a little more calculus, when $f(c)$ has a positive local maximum, it has two real positive roots.

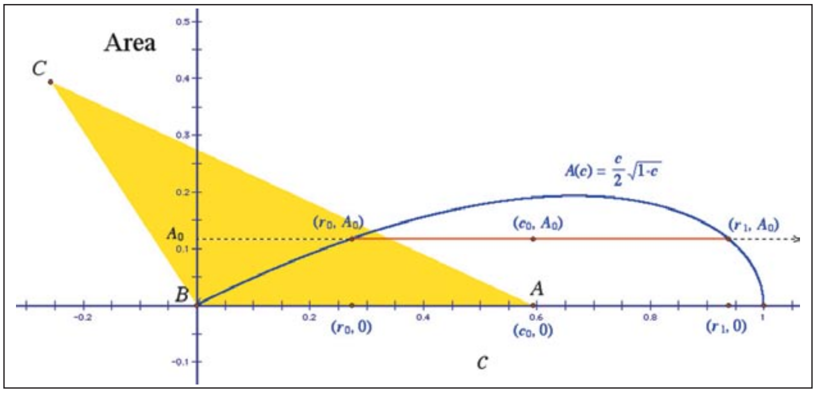

Fig. 3 The points in the region below the blue line and above the x-axis correspond to triangles with semiperimeter 1. ΔABC is the triangle corresponding to (c_0, A_0).

Editors' note: We are not sure what the authors have in mind here, but note that $f(c) = -sc^3 + s^2c^2 - 4A^2$ with $A > 0$. Since $f(0) < 0$ and $f(c)$ is negative for large values of c, if $f(2/3) > 0$, the graph of f has to cross the c-axis twice—once between 0 and 2/3 and once at some value greater than 2/3.

For any c between these two roots, there exists a unique triangle with area A, side c, and semiperimeter s, since by (1.1), A, s, and c determine the pair a, b. This shows that for any A, s satisfying (1.2) with strict inequality, there are infinitely many noncongruent triangles with the same A and s. So the answer to Chuck's question is, Definitely not.

Now that we know counterexamples exist, the following exercise becomes meaningful:

Exercise 1: Show that the radius of the incircle (the circle inscribed in a triangle) is the same for all triangles with fixed area and perimeter.

FROM ALGEBRA BACK TO GEOMETRY

Although we solved our original problem algebraically, the solution is not entirely satisfying, since we cannot explicitly find the positive roots of the evil cubic $f(c)$ (hence its name). To get a better qualitative understanding of the geometry, we need to keep track of all possible triangles at once.

As a first step, we realized that in the original problem we might as well assume that $s = 1$, as an infinite family of triangles with fixed A, s can be scaled to an infinite family with corresponding constants A/s^2, 1. This means we need to keep track of all possible triangles only up to similarity.

So, from now on, set $s = 1$. Let the area A be variable. For fixed c, yet another calculus exercise shows that the maximum value of A is

$$A = \frac{c}{2}\sqrt{1-c}. \qquad (2.1)$$

Editors' note: We let A be variable and expressed A^2 as a function, say g, of a, using Heron's formula:

$$\begin{aligned} A^2 = g(a) &= (1-c)(1-a)(1-(2-a-c)) \\ &= (1-c)(1-a)(-1+a+c) \\ &= (1-c)(-1+c+a(2-c)-a^2) \end{aligned}$$

Now, c is a constant, so this is a quadratic in a. Using either the theory of quadratic functions or calculus, we find that the maximum value of g is at

$$a = \frac{2-c}{2}.$$

And hence the maximum value produced by g is

$$g\left(\frac{2-c}{2}\right) = \left(\frac{c}{2}\right)^2 (1-c).$$

Since g's outputs are the values of A^2, the maximum value for A is

$$\frac{c}{2}\sqrt{1-c}.$$

Therefore, all triangles $s = 1$ are represented by a point (c_0, A_0) in the first quadrant of the (c, A)-plane that lie in the region $\Re$ on or under the boundary curve (2.1). (See **fig. 3**.)

The evil cubic has a pretty interpretation in this picture: If a horizontal line cuts the boundary curve at (r_0, A_0), (r_1, A_0), then r_0, r_1 are the two roots of the evil cubic

$$f(c) = -c^3 + c^2 - 4A_0^2.$$

Editors' note: This took a while to figure out, and we are not sure why. Suppose we fix a value for A, say A_0, and suppose r is a value of c on the boundary of the curve

$$A = \frac{c}{2}\sqrt{1-c}$$

so that

$$A_0 = \frac{r}{2}\sqrt{1-r}.$$

Square both sides and simplify to obtain

$$4A_0^2 = r^2 - r^3.$$

So, r is a root of the equation that defines the evil cubic:

$$0 = -4A_0^2 + r^2 - r^3$$

Thus, each point on the horizontal line inside $\Re$ corresponds to a triangle with fixed area and $s = 1$.

Exercise 2

(i) Which point in this region corresponds to the unique equilateral triangle with $s = 1$?

(ii) In **figure 3**, find the coordinates of point C, given c_0 and A_0.

Using GSP, we noticed that as we move along a horizontal line in our region, each non-isosceles triangle appears three times, as each side of a triangle can be considered to be the c side, the side on the x-axis. For example, the longest side of the three congruent triangles in **figure 4** occurs once on the x-axis, once as the left side, and once as the right side. The "side of a fixed triangle which gets to be c" changes when either $c = a$ or $c = b$ (why?), which by Heron's formula occurs when

$$A = (1 - c)\sqrt{2c - 1}. \qquad (2.2)$$

Editors' note: The side lengths are a, c, and $2 - a - c$. The area is given by

$$A^2 = (1 - c)(1 - a)(2 - a - c). \; (*)$$

Two sides are congruent if $a = c$, if $a = 2 - a - c$, or if $c = 2 - a - c$—that is, if $a = c$, if $a = (2 - c)/2$, or if $a = 2 - 2c$. In each of these three cases, substitute the expression for a into (*) and simplify.

Graphing (2.2) on the same plane as (2.1) divides $\mathfrak{R}$ into three regions I, II, III (see **fig. 5**.) Note that all isosceles triangles correspond to points on the graph of either (2.1) or (2.2).

For each region, there is a one-to-one correspondence between points of the region (including the boundary curves) and triangles with $s = 1$. In particular, no two points of a fixed region correspond to congruent or even similar triangles. Mathematicians would call any of I, II, or III a *moduli space* of triangles, as the set of all points in the region corresponds in a one-to-one way to the set of all triangles up to scaling. It may seem that any of the three regions is as good as any other, but exercise 3(ii) shows that this is not the case.

Exercise 3

(i) Take a vertical line lying inside the region $\mathfrak{R}$. Show that the corresponding set of triangles with $A = (c, 0)$, $B = (0, 0)$ as in **figure 3** have their third vertices C all lying on a common ellipse.

(ii) Find the curve in region I that corresponds to all right triangles with $s = 1$. Do the same for regions II, III. Which moduli space do you prefer?

Editors' note: Suppose that the legs of the triangle have lengths a and c and we express the area in terms of c, so that we are working in region I. Then the area is $A = (1/2)ac$, and Pythagoras gives the hypotenuse as

$$\sqrt{a^2 + c^2}.$$

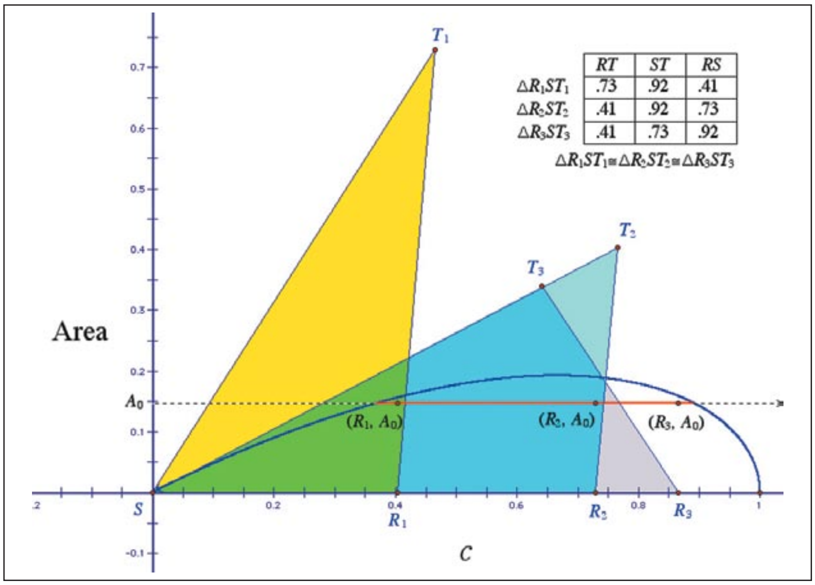

Fig. 4 Each scalene triangle with semiperimeter 1 corresponds to three points in $\mathfrak{R}$.

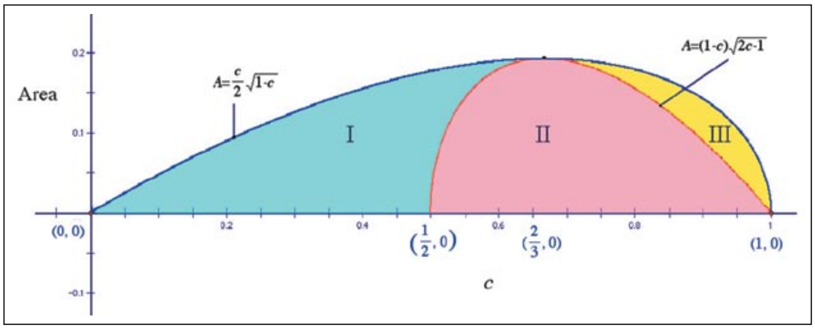

Fig. 5 Points in any of the regions I, II, or III are in one-to-one correspondence with the set of triangles with semiperimeter 1.

But the perimeter is 2, so

$$\sqrt{a^2 + c^2} = 2 - (a + c).$$

Square both sides and solve for a:

$$a = \frac{2(c-1)}{c-2}$$

so

$$A = \frac{1}{2}ac = \frac{c^2 - c}{c - 2}.$$

Graph this in region I. Regions II and III are handled similarly, with a or the hypotenuse as base. You might want to use a Computer Algebra System (CAS), as some calculations get quite messy.

(iii) Another choice of moduli space for triangles with $s = 1$ is the set $\{(a, b) : a + b > 1, a < 1, b < 1\}$, where a, b represent two sides of a triangle (and the third side is $c = 2 - a - b$). Use GSP to draw the set of all triangles of fixed area A in this moduli space. This should convince you that different moduli spaces are better for different questions about triangles.

FROM GEOMETRY TO NUMBER THEORY

The right triangles with side lengths (3, 4, 5), (5, 12, 13), and so forth are easy to work with in that they have integer sides and integer areas. When we move on to teach the laws of cosine and sine, it is harder to find Heron triangles—that is, triangles with integer sides and areas. In this section, we will go over a procedure to produce all Heron triangles and show how this number theory problem fits with our moduli space picture.

It is well known that primitive Pythagorean triples (i.e., Heron right triangles, whose sides have no common factor greater than one) are given by $(m^2 - n^2, 2mn, m^2 + n^2)$ for m, n relatively prime integers and at least one of m, n even, up to switching the first two terms. Interestingly, all these triangles have unique ratios A/s^2, a good proposition for the reader to prove. Note that this ratio is scale free—that is, the same for similar triangles—and so is a well defined function on the moduli space. To put these in our moduli space, the $s = 1$ scaled versions are

$$\left(\frac{m^2-n^2}{m(m+n)}, \frac{2mn}{m(n+m)}, \frac{m^2+n^2}{m(n+m)}\right)$$

with areas

$$A_{m,n} = \frac{n(m-n)}{m(m+n)}.$$

If you completed exercise 3(ii), you can find these points in (your choice of) the moduli space.

Non-right Heron triangles with sides (a, b, c) scale down into our space by

$$\left(\frac{a}{\frac{a+b+c}{2}}, \frac{b}{\frac{a+b+c}{2}}, \frac{c}{\frac{a+b+c}{2}}\right)$$

or

$$\left(\frac{2a}{a+b+c}, \frac{2b}{a+b+c}, \frac{2c}{a+b+c}\right).$$

For example, (20, 13, 11) has an area of 66, so the $s = 1$ version is (20/22, 13/22, 11/22).

Note that the scaled versions of Heron triangles have rational sides and area. This is no big deal, as any triangle with rational sides and area can be scaled up to a Heron triangle. From now on, a triangle with rational sides and area will be called a rational Heron triangle.

Here is a construction of all rational Heron triangles and, hence, all Heron triangles. Start with the unit circle $x^2 + y^2 = 1$. Draw the line from (0, 1) to $(t, 0)$ (where $t > 1$). It hits the unit circle at

$$\left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right),$$

so this point has rational coordinates precisely if $t \in Q$ (Q being the set of all rational numbers). Let ψ be the corresponding angle, so

$$\cos(\psi) = \frac{2t}{t^2+1}, \ \sin(\psi) = \frac{t^2-1}{t^2+1}.$$

(See **fig. 6**.) We will write $\psi \in Q$ if $t \in Q$ (equivalently, if $\cos(\psi)$, $\sin(\psi) \in Q$), and call ψ essentially rational.

Consider ΔABC with sides a, b, c and fixed angle ψ at vertex B. (We note that A refers both to a vertex of the triangle and its area, but the meaning of A should be clear from the context.) By the law of cosines, using $b = 2 - (a + c)$, then

$$a = \frac{2(1-c)}{2-c(1+\cos(\psi))}$$

and the area A of ΔABC can be found using

$$A = \frac{1}{2}ac\sin(\psi) = \frac{c(1-c)\sin(\psi)}{2-c(1+\cos(\psi))}. \qquad (3.1)$$

Therefore,

$$c,\ \psi \in Q \Rightarrow a \in Q \Rightarrow A,\ b \in Q \Rightarrow \Delta ABC$$

is rational Heron.

Editors' note: Label the triangle as the authors describe, with side a opposite $\angle A$ and so on with ψ at B. Since $a + b + c = 2$, then $b = 2 - (a + c)$. The law of cosines says that

$$(2 - (a + c))^2 = a^2 + c^2 - 2ac\cos\psi.$$

Expand and simplify to obtain

$$a = \frac{2(1-c)}{2-c(\cos\psi + 1)}.$$

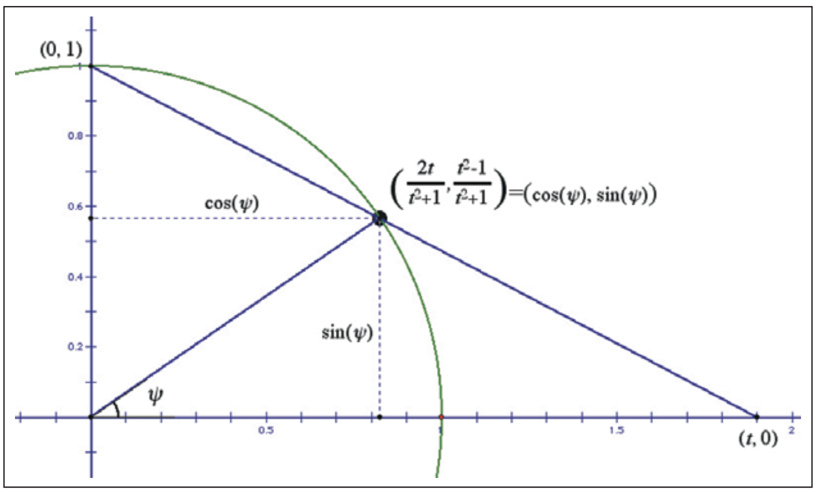

Fig. 6 The construction of all points on the unit circle with rational coordinates

Since we want to work in the (c, A)-plane, fix an essentially rational angle ψ (or, equivalently, fix a rational t). Then

$$A_\psi(c) = \frac{c(1-c)\sin(\psi)}{2-c(1+\cos(\psi))}$$

is the area of a triangle with base c and base angle ψ. (If we let ψ be arbitrary, the curves $A_\psi(c)$ sweep out the moduli space.) By calculus, $A_\psi(c)$ is increasing for c near 0. So if c runs over an infinite number of small rational numbers $c_1, c_2, \ldots$ near zero, we get an infinite number of rational Heron triangles with distinct areas corresponding to the points

$$\left(c_1,\ A_\psi(c_1)\right),\ \left(c_2,\ A_\psi(c_2)\right),\ \cdots$$

on the graph of A_ψ.

Exercise 4:
(i) Pick $c \in \mathbb{Q} \cap (0, 1)$. Show that there exists an infinite number of rational Heron triangles on the vertical line over c in the moduli space.
(ii) Show that the set of Heron triangles is dense in the moduli space—that is, show that for any circle of radius $\varepsilon > 0$ with center (c_0, A_0) in the moduli space, there exists a Heron triangle corresponding to (c, A) such that (c, A) is within the circle. Hint: Let c in 4(i) range over all elements of $\mathbb{Q} \cap (0, 1)$.

We can now produce many interesting sequences of rational Heron triangles. For example, take $t = 2$, so $\cos(\psi) = 4/5$, $\sin(\psi) = 3/5$. Using (3.1), we get the sequence

$$(a(c),\ b(c),\ c) = \left(\frac{10(1-c)}{10-9c}, \frac{-10+18c-9c^2}{10-9c},\ c\right), \quad (3.2)$$

which is equivalent to the Heron triangles $(10(-1+c),\ 10-18c+9c^2,\ c(-10+9c))$, $c \in \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of all positive integers), after clearing denominators. These triangles correspond to the points on the curve $A_\psi(c)$ in the moduli space, where $\psi = \cos^{-1}(4/5)$.

For $t = 2$, $c = 1/2$, we get the Heron triangle (20, 13, 11) after scaling. Other Heron triangles with the same angle at one vertex are obtained by listing a few rationals close to $c = 1/2$—for example, $c_1 = 1/2$, $c_2 = 7/16$, $c_3 = 17/32$, …. After clearing denominators, we get an infinite sequence of rational Heron triangles—(20, 13, 11), (1440, 985, 679), (4800, 3049, 2839), …—with corresponding areas of 66, 293328, 4088160, …. These triangles are "almost similar" to one another.

Exercise 5
Show that the set of triangles with fixed angle α at vertex C corresponds to the points on the line $A = (1-c)\tan(\alpha/2)$ in the moduli space. If $\tan(\alpha/2)$ is rational, will a and b always be rational, thus producing a rational Heron triangle?

To get other, almost similar rational Heron triangles, we could fix c and let t vary, but for fun, let's have both c and t vary. Pick, for example, $t = 2.1$, close to the original value of t, so the corresponding angle ψ has $\cos(\psi) = 4.2/5.41$, $\sin(\psi) = 3.41/5.41$. This gives a second infinite sequence of rational Heron triangles

$$\left(\frac{10.82(1-c)}{10.82-9.61c}, \frac{-10.82+19.22c-9.61c^2}{10.82-9.61c},\ c\right) \quad (3.3)$$

(cf. [3.2]). We can produce impressive Heron triangles by choosing a more complicated $c \approx 1/2$ at random, say, $c = 1623/3410$. Clearing denominators, we get the first Heron triangle in the sequence (3.3): (6593350940, 4475827709, 3456855291). A mental check shows this has area 7183142964957817770, and its sides satisfy $a^2 + c^2 - 2ac(4.2/5.41) = b^2$.

Editors' note: "Mental check," indeed.

After rescaling to $s = 1$, this triangle corresponds to a point in the moduli space very close to the point for the scaled (20, 13, 11) triangle, so these two triangles are almost similar.

VAN LUIJK'S THEOREM

Finding as many rational Heron triangles as you want on *vertical* lines in the moduli space is easy. (Go back to exercise 4(i). Hint: Fix c and vary t.) In contrast, finding rational Heron triangles on *horizontal* lines in the moduli space turns out to be a cutting-edge research problem. It was proved only in 2000 that there exists an infinite number of horizontal lines with *two* Heron triangles. The work has been expanded on since then. Here is a fairly recent preprint:

Theorem (van Luijk, www.arxiv.org/math.AG/0411606, to appear in *J. Number Theory*): There exist infinitely many rational numbers A_1, A_2, A_3, … such that for each A_i, there exists an infinite number of non-similar Heron triangles

$$\left(a_{i1}, a_{i2}, a_{i3}\right),\ \left(b_{i1}, b_{i2}, b_{i3}\right),\ \left(c_{i1}, c_{i2}, c_{i3}\right),\ \ldots \quad (4.1)$$

such that all triangles in (4.1) have the same scale-free ratio $A_i/(s_i)^2 \neq A_j/(s_j)^2$ for $i \neq j$.

Said more succinctly, van Luijk's theorem claims there are *infinitely many* rational Heron points (not just two) along any horizontal line (of rational A)

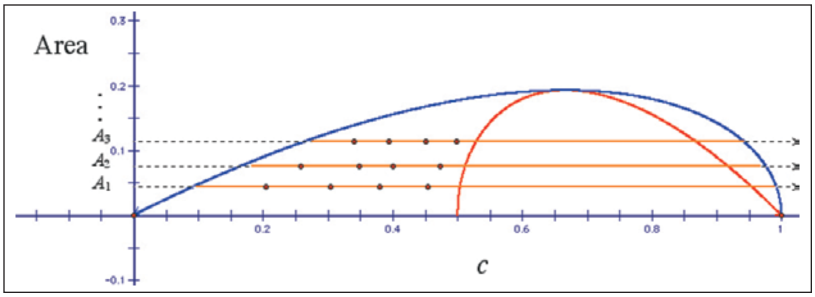

Fig. 7 The authors' interpretation of van Luijk's theorem in the moduli space

in the moduli space. (See **fig. 7**.) Van Luijk's 2004 proof is very sophisticated and uses arithmetic algebraic geometry, in particular, the theory of rational points on elliptic curves. So the number theory for horizontal lines in the moduli space is much more complicated than for vertical lines.

How hard can van Luijk's theorem be? Let's take a shot at the two-rational-Heron-triangles-per-line problem to see what we are up against.

Example: Say $\psi = \pi/2$. One triangle on

$$A_{\frac{\pi}{2}}(c)$$

is $c = 3/5$, $A = 6/7$, which corresponds to the (scaled-down) (20, 21, 29) right triangle. Let's look for another triangle with an area of 6/7. We will need a new c and a new ψ. Using the equation from the top of page 661,

$$A_{\psi}(c) = \frac{c(1-c)\sin\psi}{2-c(1+\cos\psi)} = \frac{6}{7}.$$

Substituting in the t formulas for $\sin\psi$ and $\cos\psi$, we end up with the miserable equation

$$\frac{6}{7} = \frac{c(1-c)\dfrac{t^2-1}{t^2+1}}{2-c\left(1+\dfrac{2t}{t^2+1}\right)}.$$

If we solve for this for c, the quadratic formula leaves us with an unpleasant discriminant in t. Best of luck finding a t to make the discriminant a perfect square. Alternatively, if we solve for t, we need to find the right t to make an ugly quartic work out nicely. There is no guarantee that the solution to either approach will yield the rational result desired.

While we are struggling with finding just *two* unique rational Heron triangles for one rational area, van Luijk found a way to generate infinitely many nonsimilar rational Heron triangles for *any* rational area in the moduli space.

So, van Luijk's proof of his theorem is pretty complicated. We wonder if there might be an easier approach, but we have not found it yet.

CONCLUSION

Too often the high school mathematics curriculum is compartmentalized into geometry, algebra, and number sense units. In this case, starting with an innocent question in high school geometry, we were led to a year-long discussion of algebra and moduli spaces (the geometry of the set of all triangles, or the "geometry of geometry") and to related number theory questions. In each approach, we eventually got stuck (we cannot solve the evil cubic explicitly, and we cannot reprove van Luijk's theorem), but we discovered a tremendous amount of new mathematics along the way.

There is a lot more left to uncover. For example, are there nice choices for A such that the evil cubic has three rational roots? Using t as a parameter, what triangles do you get as you climb a vertical line in the moduli space? Use GSP files to investigate other lines and curves in the moduli space, and let us know what you discover.

Final editors' note: This tour de force by Rosenberg, Spillane, and Wulf is a delightful blend of algebra, geometry, and abstraction. We welcome submissions that build on the ideas in this article. The authors suggest some directions in their conclusion. Here are some others:

- Many optimization problems can be solved without calculus. The arithmetic-geometric mean inequality is a useful tool for such purposes. The classic "Maxima and Minima without Calculus" by Ivan Niven (Dolciani Mathematical Expositions 6, Mathematical Association of America, 2005) shows just how far these techniques can take you. We would be very interested in articles around this theme.
- There have been several articles in this journal that have dealt with Heron triangles. See, for example, Bowen Kerins and the High School Teachers Program Group of the Park City Mathematics Institute, "Gauss, Pythagoras, and Heron," *Mathematics Teacher* 96, no. 5 (May 2003), pp. 350–57. There is even a connection between Heron triangles and the arithmetic of complex numbers. We are eager to receive more articles on Heron triangles.
- This whole idea of moduli space is intriguing. Where else in high school mathematics can you find interesting examples? For example, in "Regression Lines through Conic Sections" ("Delving Deeper," *Mathematics Teacher* 96, no. 9 [December 2003], pp. 634–38), we parameterize lines that are "equally as bad" with respect to a set of data by their slope and y-intercept.
- The authors also sent the following generalization for a three-dimensional application of rational points in stereographic projections:

Geometric digression

There is an important three-dimensional analogue of **figure 6**. Think of the unit sphere as the earth's surface, and let P be a point on the earth that is not the North Pole. Draw a line l_P from the North Pole through P. Call G the intersection of l_P with the xy-plane. Then the function $P \to G$ produces a map of the earth minus the North Pole. This function, called stereographic projection, still takes points with rational coordinates to points with rational coordinates; this is a good exercise. A new three-dimensional feature is that stereographic projection preserves angles. This is important for sailors, who can use the map to chart the direction of their course, although it obviously distorts distances. This trade-off is inevitable: Gauss proved that there is no distance and angle-preserving map from any piece of the sphere to any piece of the plane. ∞

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